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Radial Reflection Diffraction Tomography Notes

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Abstract

We are developing the theory behind a new imaging modality which uses a single transducer rotating about its center to launch a field radially outward and collect the backscattered (reflected) field. We use diffraction tomography techniques, based upon a linearized version of the field scattering equation, to form images of the medium surrounding the transducer. As there is one transducer which both transmits the incident field and measures the backscattered field, the operation mode is multimonostatic.

1 Introduction

Consider a single transducer which rotates about a fixed radius, R_0, launching pulses at each angular location, θ, and measuring the backscattered field. Operating in such a multimonostatic condition, we develop an imaging algorithm based upon a linearized description of the field scattering process. We summarize the operating conditions as follows:

- Transducer rotates at a fixed radius, R_0, about the origin;
- Must use frequency diversity which implies an incident pulse;
- Multimonostatic reflection operating mode in which the single transducer emits a pulse and records the backscattered field at each angular location, θ;
- Geometry is that of Figure 1.
  The transducer is located at $\mathbf{r}_0 = R_0 \left(\cos\theta, \sin\theta\right)$.
  The observation point (to be reconstructed) is located at $\mathbf{r}' = r' \left(\cos\theta', \sin\theta'\right)$ where $R_0 < r'$.

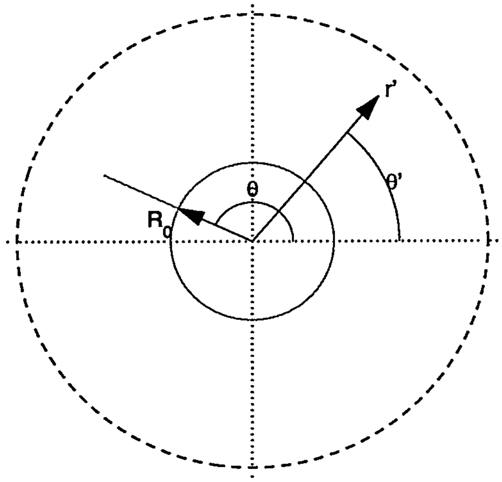

Figure 1: *Radial reflection geometry*

2 Derivation of Forward Model

Use as fundamental equation the Helmholtz equation,

$$\left[\nabla^2 + k^2(\mathbf{r})\right] u(\mathbf{r},\omega) \quad = \quad -p(\mathbf{r},\omega), \tag{1}$$

where

$\mathbf{r} \equiv (x,y) = (r,\theta)$	is the spatial coordinate,
ω	is the temporal frequency,
$k(\mathbf{r})$	is the wavenumber of the inhomogeneous medium surrounding the transducer,
$u(\mathbf{r},\omega)$	is the total field,
$p(\mathbf{r},\omega)$	is the incident pulse.

Add $k_0 u(\mathbf{r},\omega)$ to both sides of Eqn. 1 where $k_0 \equiv \omega/v_0$, and move the inhomogeneous term to the right hand side:

$$\left[\nabla^2 + k_0^2\right] u(\mathbf{r},\omega) \quad = \quad -p(\mathbf{r},\omega) - \left[k^2(\mathbf{r}) - k_0^2\right] u(\mathbf{r},\omega). \tag{2}$$

Define the *object function* as

$$o(\mathbf{r}) \quad \equiv \quad \frac{k^2(\mathbf{r})}{k_0^2} - 1, \tag{3}$$

and express Eqn. 2 as

$$\left[\nabla^2 + k_0^2\right] u(\mathbf{r},\omega) \quad = \quad -p(\mathbf{r},\omega) - k_0^2 o(\mathbf{r}) u(\mathbf{r},\omega). \tag{4}$$

We may use Green's theorem to cast the differential equation of Eqn. 4 into an integral equation,

$$u(\mathbf{r},\omega) \quad = \quad \int d\mathbf{r}'\, G(\mathbf{r}-\mathbf{r}',\omega)\, p(\mathbf{r}',\omega) + k_0^2 \int d\mathbf{r}'\, G(\mathbf{r}-\mathbf{r}',\omega)\, o(\mathbf{r}')\, u(\mathbf{r}'), \tag{5}$$

where we have ignored the boundary conditions and

$$G(\mathbf{r}-\mathbf{r}',\omega) \;=\; \frac{e^{ik_0|\mathbf{r}-\mathbf{r}'|}}{4\pi\,|\mathbf{r}-\mathbf{r}'|}. \tag{6}$$

The *primary source* is

$$u_i(\mathbf{r},\omega) \;\equiv\; \int d\mathbf{r}'\, G(\mathbf{r}-\mathbf{r}',\omega)\, p(\mathbf{r}',\omega), \tag{7}$$

so that Eqn. 5 reads

$$u(\mathbf{r},\omega) \;=\; u_i(\mathbf{r},\omega) + k_0^2 \int d\mathbf{r}'\, G(\mathbf{r}-\mathbf{r}',\omega)\, o(\mathbf{r}')\, u(\mathbf{r}',\omega). \tag{8}$$

The *scattered field* is defined as

$$u_s(\mathbf{r},\omega) \;\equiv\; u(\mathbf{r},\omega) - u_i(\mathbf{r},\omega) \;=\; k_0^2 \int d\mathbf{r}'\, G(\mathbf{r}-\mathbf{r}',\omega)\, o(\mathbf{r}')\, u(\mathbf{r}',\omega).$$

Evaluate it on the measurement surface, $r_0 = (R_0, \theta)$,

$$u_s(\theta,\omega) \;=\; k_0^2 \int d\mathbf{r}'\, G(\mathbf{r}_0-\mathbf{r}',\omega)\, o(\mathbf{r}')\, u(\mathbf{r}',\omega). \tag{9}$$

Let the incident field be a point source located at $\mathbf{r}_0$, obeying

$$\left[\nabla^2 + k_0^2\right] u_i(\mathbf{r},\omega) \;=\; -P(\omega)\,\delta(\mathbf{r}_0-\mathbf{r}), \tag{10}$$

where $P(\omega)$ is the incident pulse amplitude. NOTE: Antenna characteristics are not modeled. The incident field is then

$$u_i(\mathbf{r},\omega) \;=\; P(\omega)\, G(\mathbf{r}_0-\mathbf{r},\omega). \tag{11}$$

As is common, invoke the Born approximation,

$$o(\mathbf{r}) \;\equiv\; \frac{k^2(\mathbf{r})}{k_0^2} - 1 \;\approx\; 0, \tag{12}$$

replace the *total field* by the *incident field*, and express Eqn. 9 as

$$\boxed{u_s(\theta,\omega) \approx u_s^B(\theta,\omega) \;\equiv\; P(\omega) k_0^2 \int d\mathbf{r}'\, G^2(\mathbf{r}_0-\mathbf{r}',\omega)\, o(\mathbf{r}')} \tag{13}$$

Eqn. 13 serves as our *forward model*. Express it using Eqn. 6:

$$u_s^B(\theta,\omega) \;=\; \frac{P(\omega) k_0^2}{(4\pi)^2} \int d\mathbf{r}'\, \frac{e^{i2k_0|\mathbf{r}_0-\mathbf{r}'|}}{|\mathbf{r}_0-\mathbf{r}'|^2}\, o(\mathbf{r}') \tag{14}$$

Define the *weighted scattered field* as

$$w(\theta, 2\omega) \equiv \frac{4\pi}{P(\omega)k_0^2} u_s^B(\theta, \omega) = \frac{1}{4\pi}\int d\mathbf{r}' \, \frac{e^{i2k_0|\mathbf{r}_0-\mathbf{r}'|}}{|\mathbf{r}_0-\mathbf{r}'|^2}\, o(\mathbf{r}'). \tag{15}$$

The division by the incident pulse spectrum is equivalent to a deconvolution in the time domain. Thus, this step performs the incident pulse deconvolution of the measured field. A more convenient expression to work with is obtained by differentiating Eqn. 15 by k_0,

$$\begin{aligned}
\frac{d}{dk_0} w(\theta, 2\omega) &= \frac{1}{4\pi}\int d\mathbf{r}' \, (i2\,|\mathbf{r}_0-\mathbf{r}'|)\, \frac{e^{i2k_0|\mathbf{r}_0-\mathbf{r}'|}}{|\mathbf{r}_0-\mathbf{r}'|^2}\, o(\mathbf{r}') \\
&= \frac{i2}{4\pi}\int d\mathbf{r}' \, \frac{e^{i2k_0|\mathbf{r}_0-\mathbf{r}'|}}{|\mathbf{r}_0-\mathbf{r}'|}\, o(\mathbf{r}'), \\
&= i2 \int d\mathbf{r}' \, G(\mathbf{r}_0-\mathbf{r}', 2\omega)\, o(\mathbf{r}').
\end{aligned} \tag{16}$$

Define

$$w'(\theta, 2\omega) \equiv \frac{d}{dk_0} w(\theta, 2\omega) = v_0 \frac{d}{d\omega} w(\theta, 2\omega). \tag{17}$$

We then have

$$-\frac{i}{2}\, w'(\theta, 2\omega) = \int d\mathbf{r}' \, G(\mathbf{r}_0-\mathbf{r}', 2\omega)\, o(\mathbf{r}'). \tag{18}$$

2.1 2.5-D Problem

In cylindrical coordinates, let the measurement surface be at $\mathbf{r}_0 \equiv (R_0, \theta, z_0)$ for R_0 fixed and $0 \leq \theta < 2\pi$. Thus, Eqn. 18 becomes

$$\begin{aligned}
-\frac{i}{2} w'(\theta, 2\omega) &= \int_0^\infty r' dr' \int_0^{2\pi} d\theta' \int_{-\infty}^\infty dz' \, G(\mathbf{r}_0-\mathbf{r}', 2\omega)\, o(r', \theta'), \\
&= \int_0^\infty r' dr' \int_0^{2\pi} d\theta' \, o(r', \theta') \int_{-\infty}^\infty dz' \, G(\mathbf{r}_0-\mathbf{r}', 2\omega).
\end{aligned} \tag{19}$$

Express $|\mathbf{r}_0-\mathbf{r}'|$ in cylindrical coordinates:

$$\begin{aligned}
|\mathbf{r}_0-\mathbf{r}'|^2 &= (x_0-x')^2 + (y_0-y')^2 + (z_0-z')^2 \\
&= (R_0\cos\theta - r'\cos\theta')^2 + (R_0\sin\theta - r'\sin\theta')^2 + (z_0-z')^2 \\
&= R_0^2 + r'^2 - 2R_0 r' \cos(\theta-\theta') + (z_0-z')^2.
\end{aligned} \tag{20}$$

Defining

$$R^2 \equiv R_0^2 + r'^2 - 2R_0 r' \cos(\theta-\theta'), \tag{21}$$

we note that

$$\int_{-\infty}^\infty dz' \, G(\mathbf{r}_0-\mathbf{r}', 2\omega) = \frac{i}{4} H_0^{(1)}(2k_0 R).$$

Thus Eqn. 19 reads

$$\begin{aligned} -\frac{i}{2} w'(\theta, 2\omega) &= \frac{i}{4} \int_0^\infty r'dr' \int_0^{2\pi} d\theta' \, o(r', \theta') \, H_0^{(1)}(2k_0 R), \\ w'(\theta, 2\omega) &= -\frac{1}{2} \int_0^\infty r'dr' \int_0^{2\pi} d\theta' \, o(r', \theta') \, H_0^{(1)}(2k_0 R). \end{aligned} \tag{22}$$

Hankel function expansion from G&R (Eqn.**8.530**):

$$\boxed{H_0^{(1)}(k_0 R) = \sum_{n=-\infty}^{\infty} J_n(k_0 R_0) H_n^{(1)}(k_0 r') \, e^{in(\theta - \theta')}} \tag{23}$$

where $R_0 < r'$. Substituting this into Eqn. 22 yields

$$w'(\theta, 2\omega) = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \int_0^\infty r'dr' \int_0^{2\pi} d\theta' \, o(r', \theta') \, J_n(2k_0 R_0) H_n^{(1)}(2k_0 r') \, e^{in(\theta - \theta')} \tag{24}$$

Fourier expand the object and weighted measured field functions as follows:

$$o_n(r') = \frac{1}{2\pi} \int_0^{2\pi} d\theta' \, o(r', \theta') \, e^{-in\theta'}, \tag{25}$$

$$o(r', \theta') = \sum_{n=-\infty}^{\infty} o_n(r') \, e^{in\theta'}. \tag{26}$$

and

$$w_n'(2\omega) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \, w(\theta, 2\omega) \, e^{-in\theta}, \tag{27}$$

$$w'(\theta, 2\omega) = \sum_{n=-\infty}^{\infty} w_n'(2\omega) \, e^{in\theta}. \tag{28}$$

Using Eqn. 27 to transform Eqn. 24, yields

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} d\theta \, w'(\theta, 2\omega) \, e^{-im\theta} &= -\frac{1}{4\pi} \sum_{n=-\infty}^{\infty} J_n(2k_0 R_0) \int_0^\infty r'dr' \, H_n^{(1)}(2k_0 r') \, \times \\ &\quad \underbrace{\int_0^{2\pi} d\theta' \, o(r', \theta') \, e^{-in\theta'}}_{\equiv 2\pi \, o_n(r')} \underbrace{\int_0^{2\pi} d\theta \, e^{i(n-m)\theta}}_{\equiv 2\pi \, \delta_{mn}} \end{aligned} \tag{29}$$

Using Eqn. 25 and

$$\frac{1}{2\pi} \int_0^{2\pi} d\theta \, e^{i(m-n)\theta} = \delta_{mn}$$

reduces Eqn. 29 to

$$w'_m(2\omega) \;=\; -\pi J_m(2k_0R_0) \int_0^\infty r'dr'\; H_m^{(1)}(2k_0r')\; o_m(r'). \tag{30}$$

Define the Bessel function normalized field derivative as

$$v'_m(2\omega) \;\equiv\; -\frac{1}{\pi J_m(2k_0R_0)}\, w'_m(2\omega) \;=\; -\frac{4v_0^3}{J_m(2k_0R_0)}\, \frac{d}{d\omega}\left[\frac{u_m(\omega)}{P(\omega)\omega^2}\right], \tag{31}$$

where we have used Eqns. 15 and 17, and express Eqn. 30 as

$$v'_m(2\omega) \;=\; \int_0^\infty r'dr'\; H_m^{(1)}(2k_0r')\; o_m(r'). \tag{32}$$

If we assume the object function, $o(r,\theta)$**, is real**, we have from Eqn. 25 that

$$o^*_{-m}(r') \;=\; o_m(r').$$

Using this and the property of Hankel functions,

$$(-1)^m\, H_{-m}^{(1)}(2k_0r) \;=\; H_m^{(1)}(2k_0r),$$

we may solve for $o_m(r)$. Define

$$\begin{aligned}
P_m(2\omega) &= v'_m(2\omega) + (-1)^m v'^*_{-m}(2\omega),\\
&= \int_0^\infty r'dr'\; H_m^{(1)}(2k_0r')\; o_m(r') + \int_0^\infty r'dr'\; (-1)^m H_{-m}^{(1)*}(2k_0r')\; o^*_{-m}(r'),\\
&= \int_0^\infty r'dr'\; \left[H_m^{(1)}(2k_0r') + H_m^{(1)*}(2k_0r')\right]\; o_m(r'),\\
&= 2\int_0^\infty r'dr'\; J_m(2k_0r')\; o_m(r').
\end{aligned} \tag{33}$$

Eqn. 33 is proportional to the Bessel transform of the object,

$$o_m(2k_0) \;\equiv\; \int_0^\infty r'dr'\; J_m(2k_0r')\; o_m(r'), \tag{34}$$

so, we may express it as

$$o_m(2k_0) \;=\; \frac{1}{2}P_m(2\omega). \tag{35}$$

Explicitly expressing Eqn. 35 in terms of the measured field, we find

$$\begin{aligned}
P_m(2\omega) &= v'_m(2\omega) + (-1)^m v'^*_{-m}(2\omega),\\
&= -4v_0^3\left[\frac{1}{J_m(2k_0R_0)}\,\frac{d}{d\omega}\left[\frac{u_m(\omega)}{P(\omega)\omega^2}\right] + \frac{(-1)^m}{J_{-m}(2k_0R_0)}\,\left(\frac{d}{d\omega}\left[\frac{u_{-m}(\omega)}{P(\omega)\omega^2}\right]\right)^*\right],\\
&= -\frac{4v_0^3}{J_m(2k_0R_0)}\,\frac{d}{d\omega}\left[\frac{1}{\omega^2}\left(\frac{u_m(\omega)}{P(\omega)} + \frac{u^*_{-m}(\omega)}{P^*(\omega)}\right)\right].
\end{aligned} \tag{36}$$

Substituting Eqn. 36 into Eqn. 35, we find the "Fourier-Bessel Diffraction Theorem" for RRDT:

$$\boxed{o_m(2k_0) = -\frac{2v_0^3}{J_m(2k_0R_0)}\frac{d}{d\omega}\left[\frac{1}{\omega^2}\left(\frac{u_m(\omega)}{P(\omega)}+\frac{u^*_{-m}(\omega)}{P^*(\omega)}\right)\right]. \qquad (37)}$$

Using the orthogonality of Bessel functions, consider the following integral

$$I = \int_0^\infty kdk\, J_m(kr)\, J_m(kr') = \frac{1}{r}\,\delta(r-r'),$$

we may invert this equation for $o_m(r)$:

$$\begin{aligned}
4\int_0^\infty k_0dk_0\, P_m(2k_0v_0)\, J_m(2k_0r) &= 8\int_0^\infty k_0dk_0 \int_0^\infty r'dr' J_m(2k_0r')J_m(2k_0r)o_m(r'),\\
&= 8\int_0^\infty r'dr' \int_0^\infty k_0dk_0 J_m(2k_0r')J_m(2k_0r)o_m(r'),\\
&= 8\int_0^\infty r'dr' \frac{1}{r'}\delta(r-r')o_m(r'),\\
&= 8o_m(r).
\end{aligned}$$

$$o_m(r) = 2\int_0^\infty k_0dk_0\, P_m(2k_0)\, J_m(2k_0r). \qquad (38)$$

Substituting this into Eqn. 26 yields the reconstruction:

$$o(r,\theta) = \sum_{m=-\infty}^{\infty} e^{im\theta}\int_0^\infty k_0dk_0\, P_m(2k_0)\, J_m(2k_0r). \qquad (39)$$

Substitute Eqn. 36 into Eqn. 39:

$$\begin{aligned}
o(r,\theta) &= -8v_0^3\sum_{m=-\infty}^{\infty} e^{im\theta}\int_0^\infty k_0dk_0\,\frac{d}{d\omega}\left[\frac{1}{\omega^2}\left(\frac{u_m(\omega)}{P(\omega)}+\frac{u^*_{-m}(\omega)}{P^*(\omega)}\right)\right]\frac{J_m(2k_0r)}{J_m(2k_0R_0)},\\
&= -8v_0\sum_{m=-\infty}^{\infty} e^{im\theta}\int_0^\infty \omega d\omega\,\frac{d}{d\omega}\left[\frac{1}{\omega^2}\left(\frac{u_m(\omega)}{P(\omega)}+\frac{u^*_{-m}(\omega)}{P^*(\omega)}\right)\right]\frac{J_m(2k_0r)}{J_m(2k_0R_0)}. \qquad (40)
\end{aligned}$$

For later on, define the integral in Eqn. 40 as

$$I_m(r) \equiv \int_0^\infty \omega d\omega\,\frac{d}{d\omega}\left[\frac{1}{\omega^2}\left(\frac{u_m(\omega)}{P(\omega)}+\frac{u^*_{-m}(\omega)}{P^*(\omega)}\right)\right]\frac{J_m(2k_0r)}{J_m(2k_0R_0)}, \qquad (41)$$

and express Eqn. 40 as

$$o(r,\theta) = -8v_0^3\sum_{m=-\infty}^{\infty} I_m(r)\, e^{im\theta}. \qquad (42)$$

We must now consider how to perform numerically the integral in Eqn. 41. There is a potential problem with the $J_m^{-1}(2k_0R_0)$ term which contains an infinite number of zeros along the real axis.

2.2 Extend the Integral of Eqn. 41 over the Entire Real Axis

First some observations. The Fourier transform of the pulse is

$$P(\omega) = \int_0^\infty dt\, p(t)\, e^{i\omega t},$$

thus for a real pulse, we have

$$P(-\omega) = P^*(\omega). \tag{43}$$

The polar Fourier transform of the measured field is

$$u_m(\omega) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^\infty dt\, u(\theta, t)\, e^{i\omega t}\, e^{-im\theta},$$

thus we have

$$u_m(-\omega) = u^*_{-m}(\omega). \tag{44}$$

Using these identities, we may express $I_m(r)$ as

$$I_m(r) = \int_{-\infty}^0 \omega d\omega \frac{d}{d\omega}\left[\frac{1}{\omega^2}\left(\frac{u_m(\omega)}{P(\omega)} + \frac{u^*_{-m}(\omega)}{P^*(\omega)}\right)\right] \frac{J_m(2k_0 r)}{J_m(2k_0 R_0)}. \tag{45}$$

Adding Eqns 41 and 45, we have

$$I_m(r) = \frac{1}{2}\int_{-\infty}^\infty \omega d\omega \frac{d}{d\omega}\left[\frac{1}{\omega^2}\left(\frac{u_m(\omega)}{P(\omega)} + \frac{u^*_{-m}(\omega)}{P^*(\omega)}\right)\right] \frac{J_m(2k_0 r)}{J_m(2k_0 R_0)}, \tag{46}$$

which has extended the integral over the entire real axis.

2.3 Performing the Integral of Eqn. 46

The difficulty in integrating Eqn. 46 is the reciprocal of the Bessel function, $J_m(2k_0 R_0)$, which has multiple zeros along the real axis. We circumvent this issue by continuing the integral into the complex ω-plane and performing a contour integrtion. Care must be taken, however, since $J_m(2k_0 r)$ diverges in both the upper and lower half planes requiring us to express it as

$$J_m(2k_0 r) = \frac{1}{2}\left[H_m^{(1)}(2k_0 r) + H_m^{(2)}(2k_0 r)\right]. \tag{47}$$

and separating the integral into two. The one containing $H_m^{(1)}(2k_0 r)$ is closed in the upper half plane, whereas the one containing $H_m^{(2)}(2k_0 r)$ is closed in the lower half plane. We have

$$\begin{aligned} I_m(r) = & \frac{1}{4}\int_{C^+} \omega d\omega \frac{d}{d\omega}\left[\frac{1}{\omega^2}\left(\frac{u_m(\omega)}{P(\omega)} + \frac{u^*_{-m}(\omega)}{P^*(\omega)}\right)\right] \frac{H_m^{(1)}(2k_0 r)}{J_m(2k_0 R_0)} + \\ & \frac{1}{4}\int_{C^-} \omega d\omega \frac{d}{d\omega}\left[\frac{1}{\omega^2}\left(\frac{u_m(\omega)}{P(\omega)} + \frac{u^*_{-m}(\omega)}{P^*(\omega)}\right)\right] \frac{H_m^{(2)}(2k_0 r)}{J_m(2k_0 R_0)}, \end{aligned} \tag{48}$$

where the C^+ contour is closed in the upper half plane, and the C^- contour is closed in the lower half plane. The C^+ contour excludes the real $\omega < 0$ poles but includes the real $\omega > 0$ poles. The C^- contour is the opposite. These are shown in Figure 2.

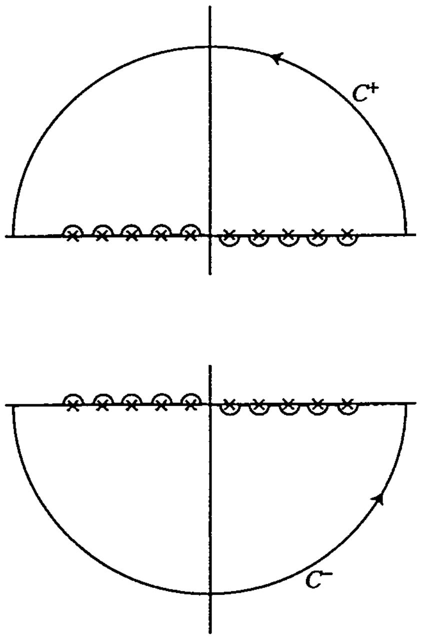

Figure 2: *Integration contours.*

2.4 The Residues at the zeros of $J_m(2k_0R_0)$

In anticipation of requiring the residues of $1/J_m(2k_0R_0)$ at the zeros of the Bessel function, we compute them as follows,

$$
\begin{aligned}
j_{mn} = \operatorname*{Res}_{z\to\alpha_{mn}}\left\{\frac{1}{J_m(z)}\right\} &= \lim_{z\to\alpha_{mn}}\frac{z-\alpha_{mn}}{J_m(z)}, \\
&= \lim_{z\to\alpha_{mn}}\frac{2}{J_{m-1}(z)-J_{m+1}(z)}, \qquad \text{(L'Hôpital's Rule)} \\
&= \frac{2}{J_{m-1}(\alpha_{mn})-J_{m+1}(\alpha_{mn})},
\end{aligned}
\tag{49}
$$

where α_{mn} is the n^{th} zero of the m^{th} Bessel function, and j_{mn} is the residue at the positive pole. For negative poles, we have

$$
\begin{aligned}
k_{mn} = \operatorname*{Res}_{z\to-\alpha_{mn}}\left\{\frac{1}{J_m(z)}\right\} &= \lim_{z\to-\alpha_{mn}}\frac{z+\alpha_{mn}}{J_m(z)}, \\
&= \lim_{z\to-\alpha_{mn}}\frac{2}{J_{m-1}(z)-J_{m+1}(z)}, \qquad \text{(L'Hôpital's Rule)} \\
&= \frac{2}{J_{m-1}(-\alpha_{mn})-J_{m+1}(-\alpha_{mn})}.
\end{aligned}
\tag{50}
$$

Use $J_m(-z) = (-1)^m J_m(z)$ to find the k_{mn} and j_{mn} are related via

$$\begin{aligned}
k_{mn} &= \frac{2}{(-1)^{m-1} J_{m-1}(\alpha_{mn}) - (-1)^{m+1} J_{m+1}(\alpha_{mn})}, \\
&= \frac{(-1)^{m+1} 2}{J_{m-1}(\alpha_{mn}) - J_{m+1}(\alpha_{mn})}, \\
&= (-1)^{m+1} j_{mn}.
\end{aligned} \tag{51}$$

2.5 Perform Complex Integration

Replace the contour integrals of Eqn. 48 by the sum of the residues,

$$\begin{aligned}
I_m(r) &= i\pi \sum_{n=1}^{\infty} \operatorname*{Res}_{2k_0R_0 \to \alpha_{mn}} \left\{ \omega \frac{d}{d\omega} \left[\frac{1}{\omega^2} \left(\frac{u_m(\omega)}{P(\omega)} + \frac{u^*_{-m}(\omega)}{P^*(\omega)} \right) \right] \frac{H_m^{(1)}(2k_0 r)}{J_m(2k_0R_0)} \right\} \\
&\quad - \frac{i\pi}{2} \sum_{n=1}^{\infty} \operatorname*{Res}_{2k_0R_0 \to -\alpha_{mn}} \left\{ \omega \frac{d}{d\omega} \left[\frac{1}{\omega^2} \left(\frac{u_m(\omega)}{P(\omega)} + \frac{u^*_{-m}(\omega)}{P^*(\omega)} \right) \right] \frac{H_m^{(1)}(2k_0 r)}{J_m(2k_0R_0)} \right\} \\
&\quad - \frac{i\pi}{2} \sum_{n=1}^{\infty} \operatorname*{Res}_{2k_0R_0 \to \alpha_{mn}} \left\{ \omega \frac{d}{d\omega} \left[\frac{1}{\omega^2} \left(\frac{u_m(\omega)}{P(\omega)} + \frac{u^*_{-m}(\omega)}{P^*(\omega)} \right) \right] \frac{H_m^{(2)}(2k_0 r)}{J_m(2k_0R_0)} \right\} \\
&\quad + i\pi \sum_{n=1}^{\infty} \operatorname*{Res}_{2k_0R_0 \to -\alpha_{mn}} \left\{ \omega \frac{d}{d\omega} \left[\frac{1}{\omega^2} \left(\frac{u_m(\omega)}{P(\omega)} + \frac{u^*_{-m}(\omega)}{P^*(\omega)} \right) \right] \frac{H_m^{(2)}(2k_0 r)}{J_m(2k_0R_0)} \right\}
\end{aligned} \tag{52}$$

Note

$$\begin{aligned}
2k_0R_0 &\to \alpha_{mn} \Rightarrow k_0 = \frac{\alpha_{mn}}{2R_0} \quad \text{and} \quad \omega = \frac{\alpha_{mn} v_0}{2R_0}, \\
2k_0R_0 &\to -\alpha_{mn} \Rightarrow k_0 = -\frac{\alpha_{mn}}{2R_0} \quad \text{and} \quad \omega = -\frac{\alpha_{mn} v_0}{2R_0}.
\end{aligned} \tag{53}$$

Define

$$\omega_{mn} \equiv \frac{\alpha_{mn} v_0}{2R_0}, \tag{54}$$

$$\hat{r} \equiv \frac{r}{R_0}, \tag{55}$$

$$A_m(\omega) \equiv \frac{d}{d\omega} \left[\frac{1}{\omega^2} \left(\frac{u_m(\omega)}{P(\omega)} + \frac{u^*_{-m}(\omega)}{P^*(\omega)} \right) \right]. \tag{56}$$

Substitute Eqns. 49 and 51 into Eqn. 52 and use the above definitions,

$$\begin{aligned} I_m(r) &= i\pi \sum_{n=1}^{\infty} \jmath_{mn}\omega_{mn}\, A_m(\omega_{mn})\, H_m^{(1)}(\alpha_{mn}\hat{r}) \\ &\quad -\frac{i\pi}{2} \sum_{n=1}^{\infty} (-1)^{m+1} \jmath_{mn}\omega_{mn}\, A_m(-\omega_{mn})\, H_m^{(1)}(-\alpha_{mn}\hat{r}) \\ &\quad -\frac{i\pi}{2} \sum_{n=1}^{\infty} \jmath_{mn}\omega_{mn}\, A_m(\omega_{mn})\, H_m^{(2)}(\alpha_{mn}\hat{r}) \\ &\quad +i\pi \sum_{n=1}^{\infty} (-1)^{m+1} \jmath_{mn}\omega_{mn}\, A_m(-\omega_{mn})\, H_m^{(2)}(-\alpha_{mn}\hat{r}) \end{aligned} \tag{57}$$

Note, using $H_m^{(1,2)}(-z) = (-1)^m H_m^{(1,2)}(z)$, we have

$$(-1)^{m+1} H_m^{(1,2)}(-z) \;=\; (-1)^{2m+1} H_m^{(1,2)}(z) \;=\; -H_m^{(1,2)}(z).$$

So we have

$$\begin{aligned} I_m(r) &= i\pi \sum_{n=1}^{\infty} \jmath_{mn}\omega_{mn}\, A_m(\omega_{mn})\, H_m^{(1)}(\alpha_{mn}\hat{r}) \\ &\quad +\frac{i\pi}{2} \sum_{n=1}^{\infty} \jmath_{mn}\omega_{mn}\, A_m(-\omega_{mn})\, H_m^{(1)}(\alpha_{mn}\hat{r}) \\ &\quad -\frac{i\pi}{2} \sum_{n=1}^{\infty} \jmath_{mn}\omega_{mn}\, A_m(\omega_{mn})\, H_m^{(2)}(\alpha_{mn}\hat{r}) \\ &\quad -i\pi \sum_{n=1}^{\infty} \jmath_{mn}\omega_{mn}\, A_m(-\omega_{mn})\, H_m^{(2)}(\alpha_{mn}\hat{r}), \\ &= i\pi \sum_{n=1}^{\infty} \jmath_{mn}\omega_{mn} \left(A_m(\omega_{mn}) + \frac{1}{2} A_m(-\omega_{mn})\right) H_m^{(1)}(\alpha_{mn}\hat{r}) \\ &\quad -\left(\frac{1}{2} A_m(\omega_{mn}) + A_m(-\omega_{mn})\right) H_m^{(2)}(\alpha_{mn}\hat{r}), \\ &= \frac{i\pi}{2} \sum_{n=1}^{\infty} \jmath_{mn}\omega_{mn} \left[\left(A_m(\omega_{mn}) - A_m(-\omega_{mn})\right) J_m(\alpha_{mn}\hat{r}) \right. \\ &\quad \left. + 3i\left(A_m(\omega_{mn}) + A_m(-\omega_{mn})\right) N_m(\alpha_{mn}\hat{r})\right]. \end{aligned} \tag{58}$$

Finally, we note that $P(\omega)$ and $u_m(\omega)$ are band limited to $\omega \in [\omega_{\min}, \omega_{\max}] \equiv \Omega$.

$$\begin{aligned} I_m(r) &= \frac{i\pi}{2} \sum_{\alpha_{mn}\in\Omega} \jmath_{mn}\omega_{mn} \left[\left(A_m(\omega_{mn}) - A_m(-\omega_{mn})\right) J_m(\alpha_{mn}\hat{r}) \right. \\ &\quad \left. + 3i\left(A_m(\omega_{mn}) + A_m(-\omega_{mn})\right) N_m(\alpha_{mn}\hat{r})\right]. \end{aligned} \tag{59}$$

Substituting Eqn. 57 back into Eqn. 40, we have the final form for the reconstruction

$$\begin{aligned} o(r,\theta) &= -4i\pi v_0 \sum_{m=-\infty}^{\infty} e^{im\theta} \sum_{\alpha_{mn}\in\Omega} j_{mn}\omega_{mn} \left[\Big(A_m(\omega_{mn}) - A_m(-\omega_{mn}) \Big) J_m(\alpha_{mn}\hat{r}) \right. \\ &\quad \left. + 3i \Big(A_m(\omega_{mn}) + A_m(-\omega_{mn}) \Big) N_m(\alpha_{mn}\hat{r}) \right]. \end{aligned} \tag{60}$$

3 Sanity Check: Test Reconstruction with a Symmetric Object

The object is independent of angle,

$$o(r,\theta) \equiv o(r).$$

Then the polar transform of the field is

$$u_m(\omega) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, u(\omega) e^{-im\theta} = u(\omega)\delta_{m0}.$$

Then Eqn. 60 reduces to

$$\begin{aligned} o(r) &= -4i\pi v_0 \sum_{\alpha_{0n}\in\Omega} j_{0n}\omega_{0n} \left[\Big(A_0(\omega_{0n}) - A_0(-\omega_{0n}) \Big) J_0(\alpha_{0n}\hat{r}) \right. \\ &\quad \left. + 3i \Big(A_0(\omega_{0n}) + A_0(-\omega_{0n}) \Big) N_0(\alpha_{0n}\hat{r}) \right]. \end{aligned} \tag{61}$$

Note,

$$A_0(\pm\omega_{0n}) \equiv \frac{d}{d\omega}\left[\frac{1}{\omega^2}\left(\frac{u(\omega)}{P(\omega)} + \frac{u^*(\omega)}{P^*(\omega)}\right)\right]_{\omega=\pm\omega_{0n}} = 2\frac{d}{d\omega}\left[\frac{1}{\omega^2}\mathrm{Re}\left\{\frac{u(\omega)}{P(\omega)}\right\}\right]_{\omega=\pm\omega_{0n}}. \tag{62}$$

Using the symmetry properties of Eqns. 43 and 44, we find

$$A_0(\omega_{0n}) - A_0(-\omega_{0n}) = 0,$$

which reduces the reconstruction to

$$o(r) = \frac{48\pi v_0^2}{R_0} \sum_{\alpha_{0n}\in\Omega} \frac{\alpha_{0n} N_0\left(\alpha_{0n}\frac{r}{R_0}\right)}{J_{-1}(\alpha_{0n}) - J_1(\alpha_{0n})} \frac{d}{d\omega}\left[\frac{1}{\omega^2}\mathrm{Re}\left\{\frac{u(\omega)}{P(\omega)}\right\}\right]_{\omega=\omega_{0n}}, \tag{63}$$

where we have substituted,

$$\omega_{0n} = \frac{\alpha_{0n} v_0}{2R_0}, \tag{64}$$

$$\hat{r} = \frac{r}{R_0}, \tag{65}$$

$$j_{0n} = \frac{2}{J_{-1}(\alpha_{0n}) - J_1(\alpha_{0n})}. \tag{66}$$

The reconstruction is pure real.

4 Fourier & Fourier-Bessel Transforms, & Fourier Diffraction Theorem

The derivations used herein are based upon the Fourier-Bessel transform pair:

$$F_m(k) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^{\infty} rdr\, f(r,\theta)\, e^{-im\theta}\, J_m(kr), \tag{67}$$

$$f(r,\theta) = \sum_{m=-\infty}^{\infty} \int_0^{\infty} kdk\, F_m(k)\, e^{im\theta}\, J_m(kr) \tag{68}$$

The Cartesian Fourier transform pair,

$$\begin{aligned} F(k_x,k_y) &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, F(x,y)\, e^{-i(k_x x + k_y y)}, \\ f(x,y) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y\, F(k_x,k_y)\, e^{i(k_x x + k_y y)}, \end{aligned}$$

can be cast into polar coordinates using the following change of variables,

$x = r\cos\theta$,	$k_x = k\cos\phi$,
$y = r\sin\theta$,	$k_y = k\sin\phi$,

to yield

$$F(k,\phi) = \int_0^{2\pi} d\theta \int_0^{\infty} rdr\, f(r,\theta)\, e^{-ikr\cos(\theta-\phi)}, \tag{69}$$

$$f(r,\theta) = \frac{1}{(2\pi)^2}\int_0^{2\pi} d\phi \int_0^{\infty} kdk\, F(k,\phi)\, e^{ikr\cos(\theta-\phi)}. \tag{70}$$

To determine the relationship between $F(k,\phi)$ and $F_m(k)$, substitute the expansion

$$e^{-ikr\cos\theta} = \sum_{m=-\infty}^{\infty} (-i)^m\, J_m(kr)\, e^{im\theta}$$

into Eqn. 69

$$\begin{aligned} F(k,\phi) &= \int_0^{2\pi} d\theta \int_0^{\infty} rdr\, f(r,\theta) \left[\sum_{m=-\infty}^{\infty} (-i)^m\, J_m(kr)\, e^{im(\phi-\theta)}\right], \\ &= \sum_{m=-\infty}^{\infty} (-i)^m\, e^{im\phi} \int_0^{2\pi} d\theta \int_0^{\infty} rdr\, f(r,\theta)\, J_m(kr)\, e^{-im\theta} \\ &= \sum_{m=-\infty}^{\infty} (-i)^m\, e^{im\phi}\, F_m(k). \end{aligned} \tag{71}$$

Using $\frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{i(m-n)\theta} = \delta_{mn}$, we may invert Eqn. 71 to find

$$F_m(k) = \frac{(-i)^{-m}}{2\pi}\int_0^{2\pi} d\phi\, F(k,\phi)\, e^{-im\phi}. \tag{72}$$

4.1 The RRDT Fourier Diffraction Theorem

Substitute Eqn. 37 into Eqn. 71 to obtain the Fourier Diffraction Theorem for RRDT,

$$\begin{aligned} O(2k_0,\phi) &= \sum_{m=-\infty}^{\infty} (-i)^m\, e^{im\phi}\, o_m(2k_0), \\ &= -2v_0^3 \sum_{m=-\infty}^{\infty} \frac{(-i)^m\, e^{im\phi}}{J_m(2k_0R_0)} \frac{d}{d\omega}\left[\frac{1}{\omega^2}\left(\frac{u_m(\omega)}{P(\omega)} + \frac{u_{-m}^*(\omega)}{P^*(\omega)}\right)\right]. \end{aligned} \tag{73}$$

This is shown graphically in Figure 3.

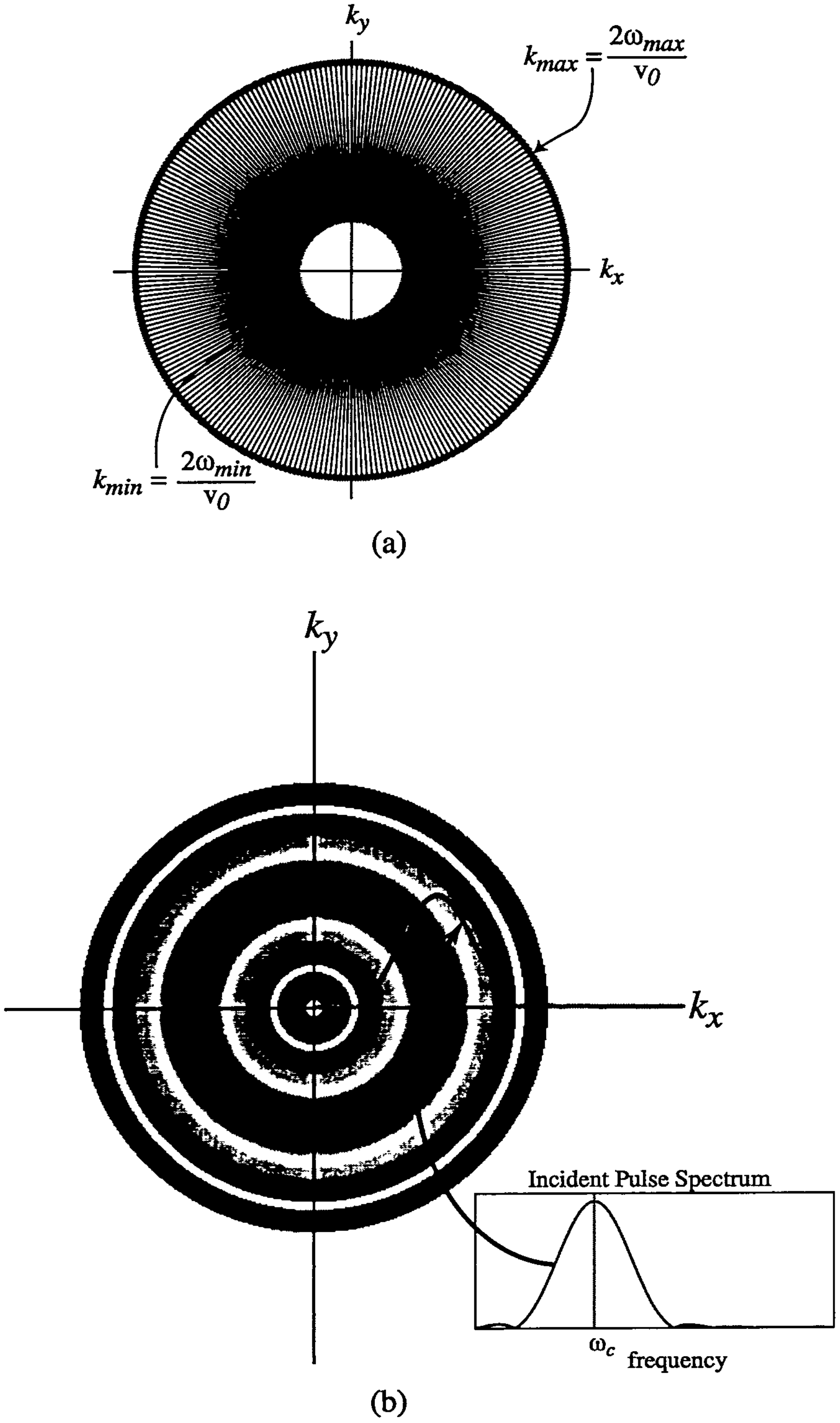

Figure 3: *Radial reflection Fourier diffraction theorem.*